Polynomial optimisation, LMI and dynamical systems

Didier HENRION
homepages.laas.fr/henrion

Prague April 2012

Outline

Measures

Measure = function assigning a number to a set

$$\mu \;:\; K \subset \mathbb{R}^n \mapsto \mathbb{R} \quad \mu(K) = \int_K d\mu = \int_K d\mu(x) = \int_K \mu(dx)$$

Examples:

- Lebesgue measure $d\mu(x) = dx$, $\mu(K) = \text{vol}(K)$
- Hermite measure $d\mu(x) = e^{-x^T x} dx$
- probability measure $\mu(K) = 1$, $\mu(K) \geq 0$
- Dirac measure $d\mu(x) = \delta_{x^*}$, $\mu(\{x^*\}) = 1$

Measures as distributions or linear functionals

Measures are a particular class of distributions, continuous linear functionals acting on test functions (infinitely differentiable functions with compact support)

Riesz representation theorems identify measures with continuous linear functionals acting on continuous functions with compact support or vanishing at infinity

So a measure can indifferently act on sets or functions

Examples:

- Lebesgue measure $f \mapsto \int_K f(x)dx$
- Hermite measure $f \mapsto \int f(x)e^{-x^T x}dx$
- probability measure $f \mapsto \int_K f(x)d\mu(x) = E[f(x)]$
- Dirac measure $f \mapsto \int f(x)\delta_{x^*} = f(x^*)$

Some terminology

Support = smallest closed set $K \subset \mathbb{R}^n$ for which $\mu(\mathbb{R}^n/K) = 0$

Examples:

- Dirac measure $\text{supp}(\delta_x) = \{x\}$
- atomic measure $\text{supp}(\mu) = \{x_1, \ldots, x_r\}$
- Hermite measure $\text{supp}(\mu) = \mathbb{R}^n$
- Lesbegue measure on $K = [-1, 1]$, $\text{vol}(K) = 2$
- Lesbegue measure on $K = \{x \in \mathbb{R}^2 \,:\, x^T x \leq 1\}$, $\text{vol}(K) = \pi$

Indicator, or characteristic function of a set K

$$\begin{aligned} I_K(x) &= 1 \quad \text{if } x \in K \\ &= 0 \quad \text{otherwise} \end{aligned}$$

Classical analysis: from functions to measures

A univariate real function

$$f : [a, b] \subset \mathbb{R} \to \mathbb{R}$$

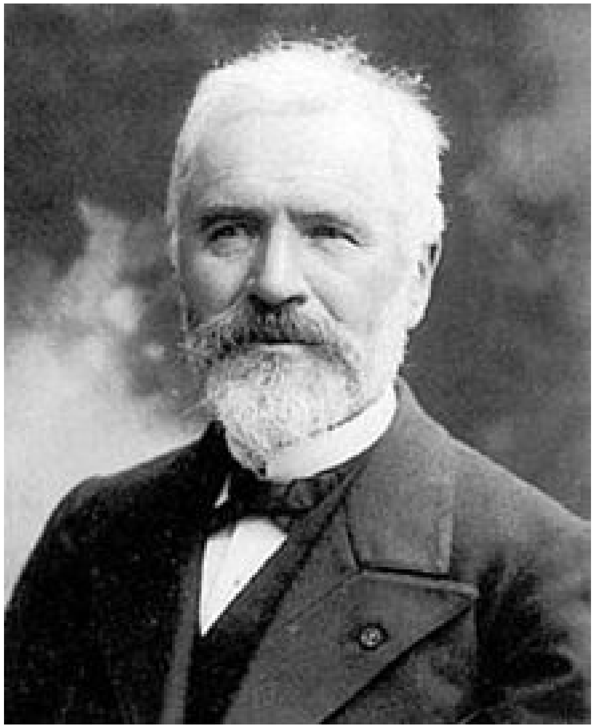
is of bounded variation (BV) whenever

$$\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|$$

stays finite over all possible partitions

$$a = x_0 < x_1 < x_2 \cdots < x_{n-1} < x_n = b$$

when $n \to \infty$



Camille Jordan
(1838-1922)

Lebesgue decomposition

Any $f(x)$ of BV can be decomposed as

$$f(x) = f_+(x) - f_-(x)$$

where f_+ and f_- are both monotone

Any $f(x)$ of BV can be decomposed as

$$f(x) = f_{AC}(x) + f_{SC}(x) + f_{SD}(x)$$

where

- f_{AC} is absolutely continuous
- f_{SC} is singular continuous
- f_{SD} is singular discrete

Henri Lebesgue
(1875-1941)

Absolutely continuous functions

Function $f(x)$ is absolutely continuous (AC) if it is continuous

$$|f(x_k) - f(x_{k-1})| \to 0 \quad \text{when} \quad |x_k - x_{k-1}| \to 0$$

and in addition

$$\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \to 0 \quad \text{when} \quad \sum_{k=1}^{n} |x_k - x_{k-1}| \to 0$$

for all possible partitions

$$a = x_0 < x_1 < x_2 \cdots < x_{n-1} < x_n = b$$

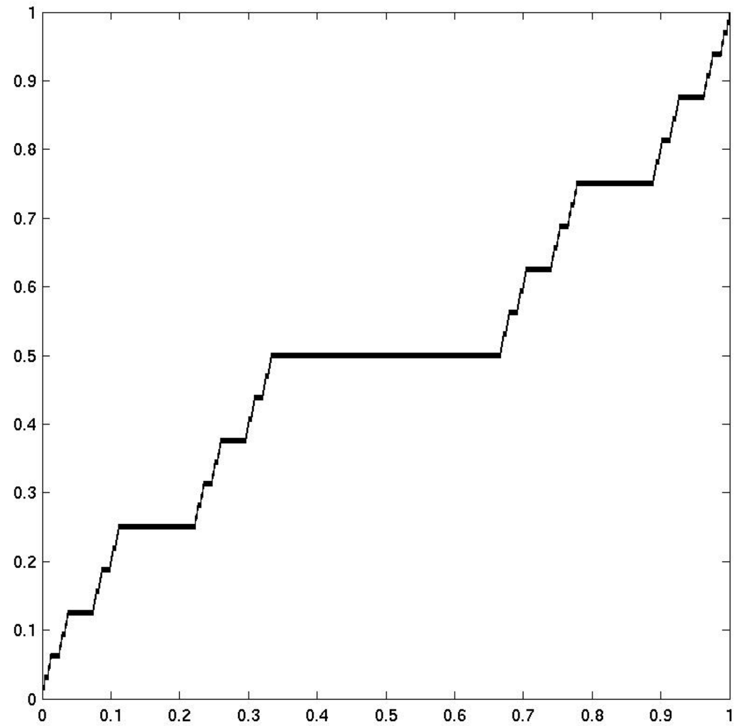
when $n \to \infty$

In particular, Lipschitz functions

$$|f(x_k) - f(x_{k-1})| \leq L|x_k - x_{k-1}|$$

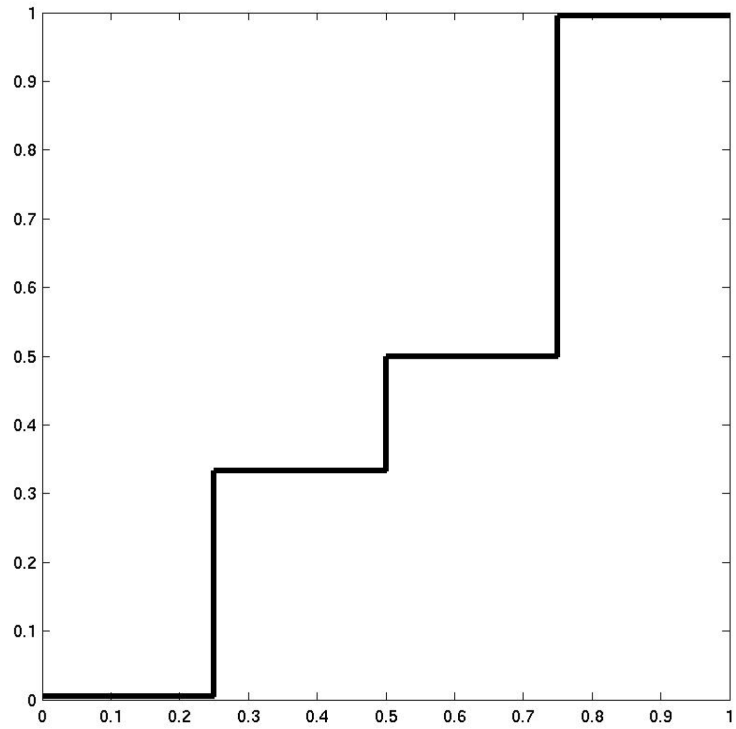
are absolutely continuous

Singular continuous functions



Cantor's devil staircase function is continuous and monotone but not absolutely continuous

Singular discrete or jump functions



Piecewise constant with discontinuities

Stieljtes integral

For $f(x)$ of BV and $v(x)$ continuous

$$\int_a^b v(x)df(x) = \lim_{|x_k - x_{k-1}| \to 0} \sum_k v(z_k)(f(x_k) - f(x_{k-1}))$$

over all possible partitions

$$a = x_0 < x_1 < x_2 \cdots < x_{n-1} < x_n = b$$

with $x_{k-1} \leq z_k \leq x_k$ when $n \to \infty$

Thomas J Stieltjes
(1856-1894)

Gives a meaning to the differential of a function of BV which is not necessarily continuous and differentiable

From functions to measures

Every continuous linear functional acting on continuous functions on $[a, b]$ can be expressed as

$$v \mapsto \int_a^b v(x) df(x)$$

with $f(x)$ a function of BV, or as

$$v \mapsto \int_a^b v(x) d\mu(x)$$

with μ a measure

Frigyes Riesz
(1880-1956)

Bijective correspondence between a function of BV and its derivative which is a generalized function = a measure

Lebesgue decomposition

By analogy with functions of BV,
any measure μ can be decomposed as

$$\mu = \mu_+ - \mu_-$$

where μ_+ and μ_- are both nonnegative measures

Any measure μ can be decomposed as

$$\mu = \mu_{AC} + \mu_{SC} + \mu_{SD}$$

where

- μ_{AC} is an absolutely continuous measure
- μ_{SC} is a singular continuous, or singular measure
- μ_{SD} is a singular discrete, or atomic measure

Moments

Multi-index notation $x^\alpha = \prod_{i=1}^n x_i^{\alpha_i}$ with $x \in \mathbb{R}^n$, $\alpha \in \mathbb{N}^n$

The α-th moment of measure μ is the real number

$$y_\alpha = \int_K x^\alpha d\mu(x)$$

μ is a representing measure for sequence $y = (y_\alpha)_{\alpha \in \mathbb{N}^n}$

Classical problem of moments (Hausdorff, Markov, Stieltjes):
characterise sequence y having representing measure μ
supported on a (given) set K

Conditions on y_α ? Construction of μ and K, given y ?

Linear operators on polynomials

Given any sequence (y_α) define the Riesz operator

$$p(x) = \sum_\alpha p_\alpha x^\alpha \in \mathbb{R}[x] \quad \mapsto \quad L_y(p) = < p, y > = \sum_\alpha p_\alpha y_\alpha \in \mathbb{R}$$

mapping polynomials to real numbers

Define the moment operator $M(y)$ such that for all $p(x) \in \mathbb{R}[x]$

$$L_y(p^2) \quad = \quad < p,\ M(y)p >$$

Given a polynomial $q(x)$, define the localizing operator $M(qy)$

$$L_y(p^2 q) \quad = \quad < p,\ M(qy)p >$$

LMI conditions

Necessary condition: if y has a representing measure μ
then $M(y) \succeq 0$

Sufficient condition (Berg 1987): if $\|y\|_\infty \leq 1$ and $M(y) \succeq 0$
then y has a representing measure μ with $\mathrm{supp}(\mu) \subset [-1, 1]$

In practice, we use finite dimensional truncations of $M(y)$
called moment matrices $M_d(y)$ of order d

The condition $M_d(y) \succeq 0$ for fixed d
is a convex linear matrix inequality (LMI) in y

Optimization over LMIs is called semidefinite programming (SDP)
for which polynomial-time interior-point methods are available

LMI conditions with support constraints

Let $K = \{x \in \mathbb{R}^n : p_k(x) \geq 0, \forall k\}$ be compact basic semialgebraic with $\{x : p_k(x) \geq 0\}$ compact for some k

Necessary condition: if y has a representing measure μ with support in K, then $M_d(y) \succeq 0$, $M_d(p_k y) \succeq 0 \ \forall k \ \forall d$

Sufficient condition (Putinar 1993): if $M_d(y) \succeq 0$, $M_d(p_k y) \succeq 0$ $\forall k \ \forall d$ then y has a representing measure with supp$(\mu) \subset K$

Here too, LMI constraints on y that can be handled with SDP

Outline

1.1. Measures, moments and LMI
1.2. Polynomial optimisation
1.3. Examples and software

2.1. Occupation measures and dynamical systems
2.2. Stability analysis
2.3. Polynomial optimal control
2.4. Examples and software

Polynomial optimisation

Consider the problem

$$\begin{array}{rcl} p^* & = & \min \;\; p(x) = \sum_\alpha p_\alpha x^\alpha \\ & & \text{s.t.} \;\; x \in K = \{x \in \mathbb{R}^n \, : \, p_k(x) \geq 0, \;\; k = 1, \ldots, m\} \end{array}$$

where unknowns are entries of vector x and K is a given basic semialgebraic set possibly nonconvex and/or nonconnected

For example K can be the union of a finite number of points, i.e. a zero-dimensional variety

Primal formulation

Linearisation

$$\begin{array}{rcll} p^* & = & \min_\mu & \int_K p(x)d\mu(x) \\ & = & \min_\mu & \sum_\alpha p_\alpha \int_K x^\alpha d\mu(x) \\ & = & \min_y & \sum_\alpha p_\alpha y_\alpha \\ & = & \min_y & L_y(p) \end{array}$$

unknowns are moments of a probability measure supported on K

Proof (lower bound):
$p(x) \geq p^*$ for all $x \in K$ so $\int_K p(x)d\mu(x) \geq \int_K p^* d\mu(x) = p^*$

Proof (upper bound):
choose a particular probability measure $\mu^* = \delta_{x^*}$ where x^* is a global minimizer, then $p^* \geq \int_K p(x)d\mu^*(x) = p(x^*)$

Hierarchy of relaxations

Use Putinar's condition to generate hierarchy of LMI relaxations

$$\begin{array}{rcl} p_d^* & = & \min_y \ L_y(p) \\ & & \text{s.t.} \ M_d(y) \succeq 0, \ M_d(p_k y) \succeq 0, \ k = 1, \ldots, m \end{array}$$

and monotonically increasing asymptotically converging sequence of lower bounds

$$p_0^* \leq p_1^* \leq \cdots p_\infty^* = p^*$$

Dual formulation

Maximize lower bound on epigraph

$$\begin{array}{rcl} p^* & = & \max \ \underline{p} \\ & & \text{s.t.} \ \ p(x) - \underline{p} \geq 0 \quad \forall x \in K \end{array}$$

involves a polynomial positivity condition which is relaxed as

$$\begin{array}{rcl} p^* & = & \max_q \ \underline{p} \\ & & \text{s.t.} \ \ p(x) - \underline{p} = (\sum_j q_{j0}^2(x)) + \sum_k (\sum_j q_{jk}^2(x)) p_k(x) \end{array}$$

with unknown polynomial sum-of-squares (SOS) multipliers

Lagrangian with polynomial multipliers

Can be formulated as a dual hierarchy of LMI problems by fixing the degree of SOS multipliers to $d = 0, 2, 4 \ldots$

Finite convergence

Unconstrained polynomial optimisation

$$\min p(x) \text{ s.t. } x \in \mathbb{R}^n, \;\; \deg p(x) = 2\delta$$

First LMI relaxation exact for $n = 1$ (univariate polynomials), $\delta = 1$ (conics), $n = \delta = 2$ (bivariate quartics)

Already known to Hilbert (1900), but first explicit counter-example of nonexactness given by Motzkin (1965) as a bivariate sextic which is nonnegative but not polynomial SOS

Artin (1927) proved however that every nonnegative polynomial is rational SOS

Finite convergence

Constrained polynomial optimisation

$$p^* = \min_x p(x) \text{ s.t. } x \in K = \{x \,:\, p_k(x) \geq 0,\ k = 1, \ldots, m\}$$

with LMI relaxations

$$\begin{array}{rcl} p_d^* & = & \min_y \ L_y(p) \\ & & \text{s.t. } \ M_d(y) \succeq 0,\ M_d(p_k y) \succeq 0,\ k = 1, \ldots, m \end{array}$$

Exactness certificate $p_d^* = p^*$ whenever

$$r = \operatorname{rank} M_d(y^*) = \operatorname{rank} M_{d-\delta}(y^*),\ \ \delta = \max_k \deg p_k(x)/2$$

moment matrix with flat extension (Curto and Fialkow 1993)

Corresponds to an r-atomic optimal measure
and we can extract minimizers x^* from $M_d(y)$

Outline

Univariate polynomials

Global minimisation of univariate polynomial

$$\min_x p(x) = \sum_{\alpha=0}^{d} p_\alpha x^\alpha, \quad x \in \mathbb{R}$$

Primal moment problem

$$\begin{array}{ll} \min_y & p_0 + \sum_{\alpha=1}^{d} p_\alpha y_\alpha \\ \text{s.t.} & H_0 + \sum_{\alpha=1}^{d} H_d y_k \succeq 0 \end{array}$$

where H_α are unit Hankel matrices

Dual SOS problem

$$\begin{array}{ll} \max_X & p_0 - \text{trace}\, H_0 X \\ \text{s.t.} & \text{trace}\, H_\alpha X = p_\alpha,\ \alpha = 1, \ldots, d \\ & X \succeq 0 \end{array}$$

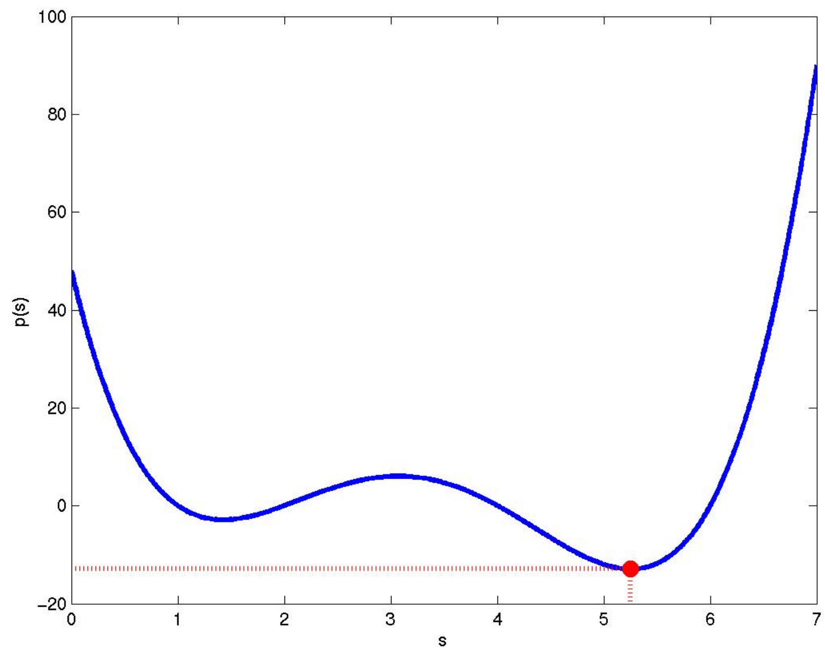
Univariate polynomials

Example: quartic polynomial

$$p(x) = 48 - 92x + 56x^2 - 13x^3 + s^4$$

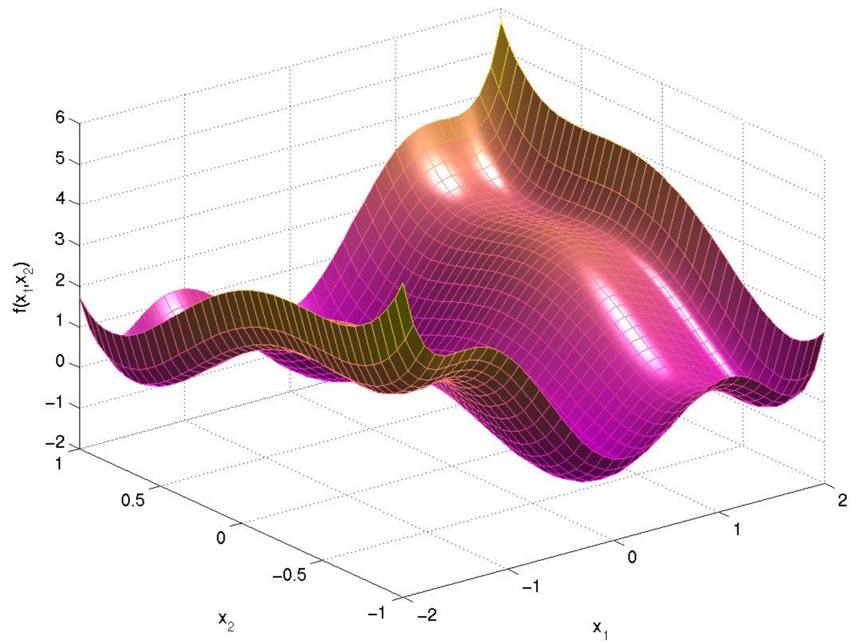
Solving the moment LMI problem yields $p^* = p(5.25) = -12.89$

$$\begin{array}{ll} \min & 48 - 92y_1 + 56y_2 - 13y_3 + y_4 \\ \text{s.t.} & \begin{bmatrix} 1 & y_1 & y_2 \\ y_1 & y_2 & y_3 \\ y_2 & y_3 & y_4 \end{bmatrix} \succeq 0 \end{array}$$



Camelback function

For the six-hump camelback function



with two global optima and six local optima, the global optimum is reached at the first LMI relaxation ($d = 1$) without any problem splitting

LMI hierarchy

Quadratic problem

$$
\begin{array}{ll}
\min & -2x_1 + x_2 - x_3 \\
\text{s.t.} & x_1(4x_1 - 4x_2 + 4x_3 - 20) + x_2(2x_2 - 2x_3 + 9) \\
& \qquad +x_3(2x_3 - 13) + 24 \geq 0 \\
& x_1 + x_2 + x_3 \leq 4, \quad 3x_2 + x_3 \leq 6 \\
& 0 \leq x_1 \leq 2, \quad 0 \leq x_2, \quad 0 \leq x_3 \leq 3
\end{array}
$$

Computational burden increases quickly with relaxation order

order d	1	2	3	4	5	6
bound p_d^*	-6.0000	-5.6923	-4.0685	-4.0000	-4.0000	-4.0000
`size(y)`	9	34	83	164	285	454

..yet fourth LMI relaxation solves globally the problem

Software for polynomial optimisation, moments and LMI

Matlab interfaces

- GloptiPoly (Henrion/Lasserre 2002)
- SOSTOOLS (Parrilo at al. 2002)
- YALMIP (Löfberg 2005)
- SparsePOP (Kojima et al. 2005)

Semidefinite programming solvers

- SeDuMi (Sturm 1999 and Terlaky 2005)
- SDPT3 (Toh et al. 1999)
- CSDP (Borchers 1999)
- SDPA (Kojima et al. 1996)
- PENSDP (Kočvara and Stingl 2004)

Outline

From Cauchy to Liouville

Dynamical system described by nonlinear ODE

$$\dot{x} = f(x)$$

with Lipschitz vector field f has a unique solution (trajectory) x_t starting from a given initial condition $x_0 \in \mathbb{R}^n$

Suppose x_0 is not known exactly: think of it as a random variable modeled by a probability measure μ_0

At time t, state x_t is also modeled by a probability measure μ_t whose time evolution is captured by a linear PDE

$$\frac{\partial \mu_t}{\partial t} + \sum_{i=1}^{n} \frac{\partial (f \mu_t)_i}{\partial x_i} = 0$$

called Liouville's transport or advection equation

Occupation measure

Given a compact set $X \subset \mathbb{R}^n$, define the occupation measure

$$\mu(X) = \int_0^T \mu_t(X)dt$$

which measures the time spent by the trajectory in X

Measure μ encodes the whole trajectory and satisfies the integral Liouville linear PDE

$$\text{div}\,(f\mu) = \mu_0 - \mu_T$$

and its variational or weak formulation

$$\int_X Dv \cdot f d\mu = \int_X v d\mu_T - \int_X v d\mu_0$$

for all smooth test functions v supported on X

Duality between measures and functions

More formally, let X be a compact topological space
Let $M(X)$ be the Banach space of finite measures
Let $C(X)$ be the Banach space of bounded continuous functions
Then $M(X)$ can be identified with the dual $C(X)^*$, in the sense that $C(X), M(X)$ form a dual pair with duality bracket

$$< v, \mu > = \int_X v d\mu$$

Let $L : C(X) \rightarrow C(Y)$ be a linear mapping
Let $L^* : M(Y) \rightarrow M(X)$ be its adjoint

$$< L(v), \mu > = < v, L^*(\mu) >$$

Duality along dynamics

Let $v(x) \in C^1(X)$ be continuously differentiable
Define linear mapping $F : C^1(X) \to C^1(X)$ such that

$$\frac{\partial v}{\partial t} = \frac{\partial v}{\partial x}\frac{dx}{dt} = Dv \cdot f = -F(v)$$

Once again, integration along system trajectories yields
linear relation linking measures μ, μ_0 and μ_T

$$\begin{array}{rcl}
-\int_X Dv \cdot f d\mu & = & \int_{X_0} v d\mu_0 - \int_{X_T} v d\mu_T \\
\int_X F(v) d\mu & = & \int_{X_0} v d\mu_0 - \int_{X_T} v d\mu_T \\
< F(v), \mu > & = & < v, d\mu_0 > - < v, d\mu_T > \\
< v, F^*(\mu) > & = & < v, d\mu_0 > - < v, d\mu_T > \\
F^*(\mu) & = & \mu_0 - \mu_T \\
\text{div}\,(f\mu) & = & \mu_0 - \mu_T
\end{array}$$

Linear measure problem

Nonlinear Cauchy problem

$$\dot{x} = f(x), \; x(0) \in X_0, \; x(T) \in X_T$$

replaced by linear problem

$$\mathrm{div}\,(f\mu) = \mu_0 - \mu_T$$

where differentiation should be understood
in the sense of distributions (Schwartz 1950)

Fluid dynamics and linear transport equation

Now suppose that $f(x)$ describes the velocity field of a compressible fluid with density $\rho(t, x)$

Time evolution of fluid density described by continuity equation from fluid dynamics, i.e. principle of mass conservation

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\,(f\rho) = 0, \quad \rho(0, x) = \rho_0(x)$$

Transportation of density of flow along trajectories

See e.g. Cédric Villani's 2003 book on optimal transport

Outline

Invariant measures

Define the Frobenius-Perron operator P, also called the push-forward of measure μ along flow x_t, as

$$P\mu(X) = \mu(x_t^{-1}(X))$$

Measure μ is called invariant if

$$P\mu(X) = \mu(X)$$

for all subsets X, i.e. if it is a fixed point of P

Invariant measures satisfy

$$\text{div}\,(f\mu) = 0$$

and they characterize stable, unstable or periodic trajectories

Equilibrium points

Let x^* satisfy $f(x^*) = 0$. Then $\mu = \delta_{x^*}$ is such that

$$\int \text{div}\,(f\mu)v = -\int Dv \cdot f\delta_{x^*} = -Dv(x^*) \cdot f(x^*) = 0$$

so it is invariant

Periodic solutions

Let $T > 0$ satisfy $x_{t+T} = x_t$ for all t
Then $\mu(X) = \frac{1}{T}\int_0^T I_X(x_t)dt$ is such that

$$\int \text{div}\,(f\mu)v = \frac{1}{T}(v(x_t) - v(x_{t+T})) = 0$$

so it is invariant

Stability

Assuming $f(0) = 0$, Rantzer (2000) observes that the existence of a density ρ such that

$$\text{div}\,(f\rho) > 0$$

implies that $x(t) \to 0$ when $t \to \infty$ for almost all $x(0)$

Dual to existence of a Lyapunov function $v(x)$ such that

$$v > 0, \;\; Dv \cdot f < 0$$

Outline

1.1. Measures, moments and LMI
1.2. Polynomial optimisation
1.3. Examples and software

2.1. Occupation measures and dynamical systems
2.2. Stability analysis
2.3. Polynomial optimal control
2.4. Examples and software

Optimal control and value function

Consider the optimal control problem (OCP)
with fixed initial condition and fixed terminal time

$$\begin{array}{rcll} V(x_0) & = & \min_{u \in U} & g(x(T)) + \int_0^T h(x,u)dt \\ & & \text{s.t.} & \dot{x} = f(x,u), \quad x(0) = x_0 \end{array}$$

Cost $x \mapsto V(x)$ is called the value function

Using calculus of variations, it can be shown that
the value function satisfies a nonlinear first order PDE..

HJB PDE

Defining the Hamiltonian

$$H(x,p) = \min_{u \in U} \{h(x,u) + p \cdot f(x,u)\}$$

the value function solves the Hamilton-Jacobi-Bellman PDE

$$H(x, DV) = 0, \quad V(x(T)) = g(x(T))$$

Under standard assumptions, the HJB PDE has a unique viscosity solution $V^* = \lim_{\varepsilon \to 0} V_\varepsilon$ with

$$H(x, DV_\varepsilon) = \varepsilon \Delta V_\varepsilon, \quad V_\varepsilon(x(T)) = g(x(T))$$

cf. P.-L. Lions (1983)

Feedback control from solution of HJB PDE

At time t for a given state $x(t)$ we let

$$u^*(x(t)) = \arg\min_u \{h(x,u) + DV^* \cdot f(x,u)\}$$

so that the Hamiltonian is minimized, i.e.

$$h(x,u^*) + DV^* \cdot f(x,u^*) = H(x, DV^*)$$

This is dynamic programming, an optimal feedback control policy

Polynomial optimal control

Consider now the OCP

$$
\begin{array}{ll}
\min_u & g(x(T)) + \int_0^T h(x,u)dt \\
\text{s.t.} & \dot{x} = f(x,u) \\
& x(0) \in X_0,\ x(T) \in X_T \\
& x \in X,\ u \in U
\end{array}
$$

with f, g, h polynomials and X_0, X_T, X, U compact basic semialgebraic sets (intersections of polynomial sublevel sets)

Weak formulation = LP on measures

Suppose μ_0 is given, the OCP can be written as a linear but infinite-dimensional problem on measures μ and μ_T

$$\begin{array}{ll} \min_{\mu,\mu_T} & \int_{X_T} g d\mu_T + \int_X h d\mu \\ \text{s.t.} & \int_X Dv \cdot f(x,u) d\mu = \int_{X_T} v d\mu_T - \int_{X_0} v d\mu_0, \quad \forall v \in C \end{array}$$

Without test functions it can be written

$$\begin{array}{ll} \min_{\mu,\mu_T} & < g, \mu_T > + < h, \mu > \\ \text{s.t.} & \mu_T + \text{div}\,(f\mu) = \mu_0 \end{array}$$

or more abstractly

$$\begin{array}{ll} \min_{\nu} & < c, \nu > \\ \text{s.t.} & < A, \nu >= b, \quad \nu \in M_+(X) \times M_+(X_T) \end{array}$$

as a primal LP on the Banach space of nonnegative measures

Dual LP

Using duality on compact Banach spaces, we obtain

$$\begin{array}{ll} \max_V & < b, V > \\ \text{s.t.} & c - < A^*, V > \in C_+(X) \times C_+(X_T) \end{array}$$

a dual LP on the space of nonnegative continuous functions that can be written explicitly as

$$\begin{array}{ll} \max_V & < \mu_0, V > = \int_{X_0} V d\mu_0 \\ \text{s.t.} & < \mu, h + DV \cdot f > = \int_X (h + DV \cdot f) d\mu \geq 0 \\ & < \mu_T, g - V > = \int_{X_T} (g - V) d\mu_T \geq 0 \end{array}$$

Conic complementarity

By conic complementarity, along optimal trajectories (x^*, u^*) and for optimal dual function V^* it holds

$$< h + DV^* \cdot f, \mu^* >=< g - V^*, \mu_T^* >= 0$$

or equivalently

$$\begin{array}{l} H(x^*, DV^*) = h(x^*, u^*) + DV^* \cdot f(x^*, u^*) = 0 \\ V^*(x^*(T)) = g(x^*(T)) \end{array}$$

which means that V^* solves the HJB PDE

Solving primal problem on measures = solving OCP

Solving dual problem on functions = solving HJB PDE

How do we proceed numerically ?

Generalized problem of moments

We face linear problems involving several measures μ_i respectively supported on semialgebraic sets X_i

All the data are polynomials, so we can replace measures by their moments (e.g. $\int_{X_i} h_i(x) d\mu_i = \int_{X_i} \sum_\alpha h_{i\alpha} x^\alpha d\mu_i = \sum_\alpha h_{i\alpha} \int_{X_i} x^\alpha d\mu_i$)

$$\begin{array}{ll} \min_\mu & \sum_i \int_{X_i} h_i d\mu_i \\ \text{s.t.} & \sum_i \int_{X_i} a_{ij} d\mu_i = b_j \\ & \text{measures } \mu_i \end{array} \qquad \begin{array}{ll} \min_y & \sum_i \sum_\alpha h_{i\alpha} y_{i\alpha} \\ \text{s.t.} & \sum_i \sum_\alpha a_{ij\alpha} y_{i\alpha} = b_j \\ & \text{moments } y_i \end{array}$$

provided we can handle the representation condition

$$y_{i\alpha} = \int_{X_i} x^\alpha d\mu_i(x)$$

Moment LP as LMI

Using Putinar's representation conditions we obtain

$$\begin{array}{ll} \min_y & c^T y \\ \text{s.t.} & Ay = b \\ & y_\alpha = \int_X x^\alpha d\mu \\ & X = \{x \,:\, p_k(x) \geq 0, \forall k\} \end{array}$$

infinite-dimensional

LP problem

$$\begin{array}{ll} \min_y & c^T y \\ \text{s.t.} & Ay = b \\ & M_d(y) \succeq 0 \\ & M_d(p_k y) \succeq 0, \forall k \end{array}$$

finite-dim. LMI

relaxation of order d

our familiar hierarchy of LMI relaxations

Compare with static polynomial optimisation: dynamics are now taken into account by introducing several measures whose moments are linearly constrained

Dual function LP as LMI

Dual to LMI moment problem yields polynomial supersolution of HJB PDE with polynomial sign conditions enforced by polynomial SOS conditions

Good approximation of value function along optimal trajectories

For example, if $f(x,u) = f_1(x) + f_2(u)$ and $h(x,u) = h_1(x) + u^T u$ use first-order optimality condition

$$\partial_u(h(x,u) + DV^* \cdot f(x,u)) = 2u + DV^* \cdot f_2 = 0$$

to derive state-feedback control law

$$u^*(x) = -\frac{1}{2} DV^*(x) \cdot f_2(x)$$

Outline

Example of linear ODE analysis

Consider the scalar linear ODE

$$\dot{x} = -x$$

with initial measure μ_0 in $X_0 = \{x \,:\, p_0(x) = \frac{1}{4} - (x - \frac{3}{2})^2 \geq 0\}$
with terminal measure μ_T in $X_T = \{x \,:\, p_T(x) = \frac{1}{4} - x^2 \geq 0\}$
with occupation measure μ in $X = \{x \,:\, p(x) = 4 - x^2 \geq 0\}$

We want to find trajectories minimising the energy $\int_0^T x^2 dt$

Linear measure problem

$$\begin{array}{rl} \min & \int_0^T x^2 d\mu(x) \\ \text{s.t.} & \int_X Dv(x)(-x)d\mu(x) = \int_{X_T} v d\mu_T - \int_{X_0} v d\mu_0, \quad \forall v \end{array}$$

Example of linear ODE analysis

Setting $v = x^\alpha$ we introduce sequences y_0, y_T, y representing measures μ_0, μ_T, μ, and we obtain the linear moment problem

$$\begin{array}{rl} \min & y_2 \\ \text{s.t.} & -\alpha y_\alpha = y_{T\alpha} - y_{0\alpha}, \quad \forall \alpha \end{array}$$

and the corresponding LMI relaxation of order d

$$\begin{array}{rl} \min & y_2 \\ \text{s.t.} & -\alpha y_\alpha = y_{T\alpha} - y_{0\alpha}, \quad \forall \alpha,\ |\alpha| \leq 2d \\ & M_d(y_0) \succeq 0,\ M_d(y_T) \succeq 0,\ M_d(y) \succeq 0 \\ & M_d(p_0 y_0) \succeq 0,\ M_d(p_T y_T) \succeq 0,\ M_d(py) \succeq 0 \end{array}$$

Solving LMI relaxations of increasing orders d yields a sequence of monotonically increasing lower bounds on the optimum

Example of linear ODE analysis

This problem can be solved analytically, with optimal trajectory $x(t) = e^{-t}$ leaving X_0 at $x(0) = 1$ and reaching X_T at $x(T) = \frac{1}{2}$ for $T = \log 2 \approx 0.6931$

Moment matrix $M(y)$ has entries $y_\alpha = \int_0^{\log 2} e^{-\alpha t} dt = \frac{1-2^{-\alpha}}{\alpha}$

We get with SeDuMi 1.1R3 the following sequence of valid significant digits on T: 0, 2, 4, 7, 10, 13 (fast convergence)

Convergence at a finite relaxation order is impossible since the optimum is transcendental, whereas the solution of an integer coefficient LMI is algebraic

Linear ODE analysis

More generally, for the first-order linear Cauchy problem

$$\dot{x} = Ax, \;\; x(0) = x_0, \;\; x(\infty) = 0$$

the moment LMI problem reads

$$AQ + (AQ)^T = Q_0 \succeq 0, \;\; Q \succeq 0$$

with Q_0, Q nonzero covariance matrices of μ_0 (initial measure) and μ (occupation measure) respectively

Infeasible if and only if dual Lyapunov LMI problem

$$A^T P + PA \prec 0, \;\; P \succ 0$$

is feasible

Nonlinear stabilization

For the nonlinear polynomial system

$$\dot{x} = f(x) + g(x)u$$

enforce stability condition div $((f + gu)\rho) > 0$ with

$$\rho(x) = \frac{p_1(x)}{p_0(x)}, \quad u(x)\rho(x) = \frac{p_2(x)}{p_0(x)}$$

for $p_0(x)$ given positive polynomial ensuring integrability
and $p_1(x)$, $p_2(x)$ polynomials to be found

Convex design LMI, rational stabilizing feedback $u(x) = \frac{p_2(x)}{p_1(x)}$

Analogy with the Benamou-Brenier fluid mechanics approach
to the Monge-Kantorovich mass transfer problem (2000)

Software

GloptiPoly 3 (DH, JB. Lasserre, J. Löfberg) for Matlab
models generalised problems of moments as LMI problems

POCP (C. Savorgnan) models polynomial optimal control
problems as generalised problems of moments

`homepages.laas.fr/henrion/software`

Can explicitly address state constraints,
impulsive controls, discontinuous trajectories..

Concluding remarks

Hierarchy of LMI relaxations (convex optimisation) for

- static nonconvex polynomial optimisation
- (polynomial approximation of) nonconvex optimal control of dynamical systems (with polynomial dynamics)

Measures can handle piecewise polynomial models, impulsive controls, hybrid dynamics..

Current topics: dynamical systems, calculus of variations (Young measures), HJB PDEs, optimal transport

Other possible topics: chaotic dynamics, invariant measures, ergodic theory, Frobenius-Perron and Koopmans operators, game theory..